AAPT UNITED STATES PHYSICS TEAM
AIP 2006

Solutions to Problems

Part A

Question 1

Sooner or later we need to find both the location of the center of mass r and the rotational inertia I about the axis of rotation. Let's do these two things first.

We can treat the top disk as a solid disk of mass M_s with a hole of mass M_h cut out. Since the radius of the hole is half the radius of the disk, we can conclude that

$$M_h = \frac{\pi (R/2)^2}{\pi R^2} M_s = \frac{1}{4} M_s. \tag{A1-1}$$

so

$$M = M_s - M_h = \frac{3}{4} M_s = 3M_h. \tag{A1-2}$$

By symmetry the center of mass will be on a line connecting the centers of both the hole and the disk, and must be located a distance r to the right of the center of the disk, where

$$M_s(0) = M(-r) + M_h \frac{R}{2}. \tag{A1-3}$$

Then

$$r = \frac{R}{6}. \tag{A1-4}$$

We can apply a similar approach to find the rotational inertia about the axis of rotation:

$$I_s = I + I_h, \tag{A1-5}$$

where I_s is the rotational inertia of a solid disk and I_h is the rotational inertia of the part that used to be in the hole, both measured about the axis of rotation. I_s is easy,

$$I_s = \frac{1}{2} M_s R^2 = \frac{2}{3} M R^2, \tag{A1-6}$$

while I_h requires an application of the parallel axis theorem,

$$I_h = \frac{1}{2} M_h \left(\frac{R}{2}\right)^2 + M_h \left(\frac{R}{2}\right)^2 = \frac{1}{8} M R^2. \tag{A1-7}$$

Finally,

$$I = I_c + I_h = \frac{13}{24}MR^2. \tag{A1-8}$$

Now consider the motion of the center of mass. It is rotating about the axis of rotation with an angular speed ω, so that the *magnitude* of the acceleration of the center of mass is given by

$$a = r\omega^2. \tag{A1-9}$$

The net force on the disk when the center of mass is at the top is then given by

$$F_{\text{net}} = Mg - F_{\text{N}} = Mr\omega^2, \tag{A1-10}$$

where F_{N} is the normal force. The disk begins to hop when the normal force is zero, so

$$\omega^2 = \frac{g}{r} = \frac{6g}{R}. \tag{A1-11}$$

The total kinetic energy is then

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{13}{24}MR^2\right)\left(\frac{6g}{R}\right) = \frac{13}{8}MgR. \tag{A1-12}$$

Question 2

a. Treat the system as two capacitors in parallel. One has a spacing of $2d$ and a surface area of $L(L-y)$, the other a spacing of $2d-d$ and a surface area of Ly. The capacitance is then found from the formula $\epsilon_0 A/d$ and the rule for capacitors in parallel to be

$$C = \frac{\epsilon_0 L(L-y)}{2d} + \frac{\epsilon_0 Ly}{d} = \frac{\epsilon_0 L(L+y)}{2d}. \tag{A2-1}$$

b. The potential energy stored in the capacitor is given by

$$U = \frac{1}{2}\frac{Q^2}{C}, \tag{A2-2}$$

so the electric force of attraction on the slab is

$$F = -\frac{\partial U}{\partial y} \tag{A2-3}$$

$$= -\frac{Q^2}{2C^2}\frac{\partial C}{\partial y} \tag{A2-4}$$

$$= \frac{dQ^2}{\epsilon_0 L(L+y)^2}. \tag{A2-5}$$

But this is equal to the force from gravity, mg, when $y = L/2$, so

$$m = \frac{dQ^2}{\epsilon_0 gL(L+L/2)^2} = \frac{4dQ^2}{9\epsilon_0 gL^3}. \tag{A2-6}$$

c. We need to expand the expression for the force about the point $y = L/2$. Letting $y = L/2 + x$, we get

$$\begin{aligned} F &= \frac{dQ^2}{\epsilon_0 L(3L/2 + x)^2}, && \text{(A2-7)}\\ &= \frac{4dQ^2}{9\epsilon_0 L^3}\left(1 + \frac{2x}{3L}\right)^{-2}, && \text{(A2-8)}\\ &\approx \frac{4dQ^2}{9\epsilon_0 L^3}\left(1 - 2\frac{2x}{3L}\right), && \text{(A2-9)}\\ &= \frac{4dQ^2}{9\epsilon_0 L^3} - \frac{16dQ^2}{27\epsilon_0 L^4}x. && \text{(A2-10)} \end{aligned}$$

We know the first term in the last line balances gravity; it is the second term that is of interest. It is useful to write it in terms of the mass,

$$\Delta F = -\frac{16dQ^2}{27\epsilon_0 L^4}x = -\frac{4mg}{3L}x, \tag{A2-11}$$

so that the effective spring constant k is given by

$$k = \frac{4mg}{3L}. \tag{A2-12}$$

The frequency of small oscillations is then

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{\pi}\sqrt{\frac{g}{3L}}. \tag{A2-13}$$

Question 3

a. First find γ:

$$\gamma = \frac{C_P}{C_V} = \frac{C_V + R}{C_V} = \frac{\frac{3}{2} + 1}{\frac{3}{2}} = \frac{5}{3}. \tag{A3-1}$$

Fast processes are adiabatic processes, especially when we are told that no heat exchange occurs! This means that pV^γ is a constant, and then

$$p_0 V_0^\gamma = p_1 V_1^\gamma. \tag{A3-2}$$

But

$$pV = nRT, \tag{A3-3}$$

so

$$pV^\gamma = \frac{nRT}{V}V^\gamma = nRTV^{\gamma-1} \tag{A3-4}$$

is a constant. Finally, we can write

$$T_0 V_0^{\gamma-1} = T_1 V_1^{\gamma-1}. \tag{A3-5}$$

Work is done on the gas by the falling cylinder *and* the external atmosphere. Hence,

$$W = mg(y_0 - y_1) + p_0(V_0 - V_1), \tag{A3-6}$$

where m is the mass of the piston. But the cross sections, A, are uniform, so we can write $V = Ay$ and then the work done is

$$W = (mg + p_0 A)(y_0 - y_1) \tag{A3-7}$$

This work must equal the change in internal energy of the gas,

$$W = \Delta U = nC_V \Delta T = nC_V T_0 \left(\frac{T_1}{T_0} - 1\right). \tag{A3-8}$$

But

$$\frac{T_1}{T_0} = \left(\frac{V_0}{V_1}\right)^{\gamma - 1}, \tag{A3-9}$$

and

$$nC_V T_0 = \frac{3}{2} nRT_0 = \frac{3}{2} p_0 V_0, \tag{A3-10}$$

so

$$(mg + p_0 A)(y_0 - y_1) = \frac{3}{2} p_0 A y_0 \left(\left(\frac{y_0}{y_1}\right)^{\gamma - 1} - 1\right), \tag{A3-11}$$

or

$$(mg + p_0 A)\left(1 - \frac{y_1}{y_0}\right) = \frac{3}{2} p_0 A \left(\left(\frac{y_0}{y_1}\right)^{\gamma - 1} - 1\right). \tag{A3-12}$$

But $y_0 = 8y_1$, so

$$(mg + p_0 A)\left(\frac{7}{8}\right) = \frac{3}{2} p_0 A \,(3). \tag{A3-13}$$

Solve for m, and

$$m = \frac{29}{7}\frac{p_0 A}{g}. \tag{A3-14}$$

We actually want the density, so

$$\rho = \frac{m}{Ah} = \frac{29}{7}\frac{p_0}{gh}. \tag{A3-15}$$

b. Eventually the gas temperature returns to T_0 and then

$$p_0 V_0 = p_2 V_2. \tag{A3-16}$$

The pressure will rise to the point that it can support the cylinder,

$$mg = A(p_2 - p_0), \tag{A3-17}$$

so

$$mg = Ap_0 \left(\frac{V_0}{V_2} - 1\right) \tag{A3-18}$$

Rewriting in terms of the ratio y_0/y_2,

$$\frac{y_0}{y_2} = \frac{mg}{p_0 A} + 1, \tag{A3-19}$$

$$= \frac{29}{7} + 1, \tag{A3-20}$$

$$= \frac{36}{7}. \tag{A3-21}$$

and then

$$\frac{y_2}{y_0} = \frac{7}{36}. \tag{A3-22}$$

Question 4

a. The ions move, for the most part, in uniform circular motion under the influence of a magnetic force, so

$$\frac{mv^2}{r} = qvB. \tag{A4-1}$$

or

$$mv = qBr. \tag{A4-2}$$

When $r = R$ we have the maximum possible momentum, so

$$p_{\max} = qBR. \tag{A4-3}$$

b. Since

$$K = \frac{p^2}{2m}, \tag{A4-4}$$

we have

$$K = \frac{q^2B^2R^2}{2m} \tag{A4-5}$$

But

$$m = qB\frac{R}{v} = \frac{qB}{\omega}, \tag{A4-6}$$

so

$$K = \frac{qBR^2\omega}{2}. \tag{A4-7}$$

c. The ions pick up this kinetic energy from the changing potential, collecting $2qV_0$ after every revolution. The number of revolutions required is then

$$N = \frac{BR^2\omega}{4V_0}, \tag{A4-8}$$

and since each revolution requires a time $2\pi/\omega$, the total time t is given by

$$t = \frac{\pi BR^2}{2V_0}. \tag{A4-9}$$

d. The cyclotron works only because the revolution of the charged particle is constant in time. $p = qBr$ is relativistically correct, but now the velocity is given by the condition that $p = \gamma mv$. The angular frequency of revolution needs to be ω in order to get a boost each half-cycle, so

$$v = r\omega \tag{A4-10}$$

Then

$$p = m\frac{v}{\sqrt{1 - v^2/c^2}} = \frac{mr\omega}{\sqrt{1 - r^2\omega^2/c^2}}. \tag{A4-11}$$

Putting this into the momentum expression yields

$$B = \frac{m\omega}{q}\left(1 - \frac{r^2\omega^2}{c^2}\right)^{-1/2} \tag{A4-12}$$

Part B

Question 1

a. For simplicity we write v_t instead of v_{term}. Wherever possible we will solve the problem in the simplest possible manner.

i. The magnitude of the force of gravity on the wire is

$$F_g = mg, \tag{B1-1}$$

the magnitude of the magnetic force on a current I in the wire is

$$F_m = BIX, \tag{B1-2}$$

the induced emf, V, in the wire when traveling with speed v is

$$V = -BvX, \tag{B1-3}$$

and, finally, the relationship between the emf and current is

$$V = IR. \tag{B1-4}$$

Terminal velocity occurs when there is no acceleration, so that the net force is zero. Then

$$mg = BIX, \tag{B1-5}$$

$$mg = B\frac{V}{R}X, \tag{B1-6}$$

$$mg = -B\frac{BvX}{R}X, \tag{B1-7}$$

$$-\frac{mgR}{B^2X^2} = v. \tag{B1-8}$$

It is traditional to give the terminal *speed*, and consequently we don't usually write the negative sign indicating downward motion. Henceforth,

$$v_t = \frac{mgR}{B^2X^2}. \tag{B1-9}$$

ii. Let the above symbols, except for v_t, now be variables in time. Define a *dot* notation for time derivatives, so that velocity is given by

$$v = \frac{dy}{dt} = \dot{y}, \tag{B1-10}$$

and

$$a = \frac{dv}{dt} = \dot{v} = \ddot{y}. \tag{B1-11}$$

The acceleration of the wire is given by Newton's second law,

$$\begin{aligned} ma &= F_m - F_g, & \text{(B1-12)}\\ m\dot{v} &= BIX - mg, & \text{(B1-13)}\\ m\dot{v} &= -B\frac{BvX}{R}X - mg, & \text{(B1-14)}\\ \dot{v} &= -\frac{B^2X^2}{mR}v - mg, & \text{(B1-15)}\\ \dot{v} &= -g\left(\frac{v}{v_t} + 1\right). & \text{(B1-16)}\\ & & \text{(B1-17)} \end{aligned}$$

We must now integrate this expression,

$$\begin{aligned} \frac{dv}{dt} &= -g\left(\frac{v}{v_t} + 1\right), & \text{(B1-18)}\\ \frac{dv}{(v + v_t)} &= -\frac{g}{v_t}\,dt, & \text{(B1-19)}\\ \int \frac{dv}{(v + v_t)} &= -\int \frac{g}{v_t}\,dt, & \text{(B1-20)}\\ \ln\left(\frac{v}{v_t} + 1\right) &= -\frac{gt}{v_t}. & \text{(B1-21)} \end{aligned}$$

Solving for v,

$$v = v_t\left(e^{-gt/v_t} - 1\right). \qquad \text{(B1-22)}$$

iii. Thermal energy dissipates at a rate given by

$$P = IV = V^2/R, \qquad \text{(B1-23)}$$

so

$$\begin{aligned} P &= \frac{B^2v^2X^2}{R}, & \text{(B1-24)}\\ &= mgv_t\left(e^{-gt/v_t} - 1\right)^2. & \text{(B1-25)} \end{aligned}$$

iv. It is tempting to integrate the above expression. Don't do it! Instead, focus on the fact that the energy dissipated is equal to the change in energy of the wire. Note that it is moving at terminal speed when it completely leaves the field region, so

$$E = mgD - \frac{1}{2}mv_t^2. \qquad \text{(B1-26)}$$

b. Be warned that we never actually need to find L in this problem, so don't spend time trying to calculate it!

i. Equations B1-1, B1-2, and B1-3 are *still* true. But now the emf is related to the inductance and the change in current by

$$V = L\dot{I}. \qquad \text{(B1-27)}$$

The dot above the I means time derivative. Combining,

$$\begin{aligned} m\dot{v} &= BIX - mg, & \text{(B1-28)} \\ m\ddot{v} &= B\dot{I}X, & \text{(B1-29)} \\ &= -B\frac{V}{L}X, & \text{(B1-30)} \\ &= -\frac{B^2X^2}{L}v. & \text{(B1-31)} \end{aligned}$$

This is the differential equation for a simple harmonic oscillator, with angular frequency ω given by

$$\omega = \sqrt{\frac{B^2X^2}{mL}}. \quad \text{(B1-32)}$$

The period of oscillation is then

$$T = \frac{2\pi}{\omega} = \frac{2\pi\sqrt{mL}}{BX}. \quad \text{(B1-33)}$$

ii. Oscillatory motion is necessarily of the form

$$y = A\cos(\omega t + \phi) + B, \quad \text{(B1-34)}$$

where A, B, and ϕ are constants. Since y is a maximum when $t = 0$, we can conclude that $\phi = 0$. The acceleration will be given by $a = \ddot{y}$, and the maximum acceleration is then $a_{\text{max}} = A\omega^2$. This maximum acceleration occurs when the object is released, and there is not yet any current through the wire, so, from Eq. B1-28,

$$\begin{aligned} mA\omega^2 &= mg, & \text{(B1-35)} \\ A &= \frac{mgL}{B^2X^2}. & \text{(B1-36)} \end{aligned}$$

Finally, $y = D$ when $t = 0$, so

$$\begin{aligned} y &= A\cos(\omega t) + D - A, & \text{(B1-37)} \\ &= \frac{mgL}{B^2X^2}\left(\cos\left(\sqrt{\frac{B^2X^2}{mL}}t\right) - 1\right) + D. & \text{(B1-38)} \end{aligned}$$

iii. Using $v_{\text{max}} = A\omega$,

$$\begin{aligned} K_{\text{max}} &= \frac{1}{2}m{v_{\text{max}}}^2, & \text{(B1-39)} \\ &= \frac{1}{2}m\left(\frac{g}{\omega}\right)^2, & \text{(B1-40)} \\ &= \frac{1}{2}\frac{g^2m^2L}{B^2X^2}. & \text{(B1-41)} \end{aligned}$$

iv. At the bottom point of the path all of the "lost" potential energy of the wire must be stored in the magnetic field, so

$$E_B = 2mgA = \frac{2m^2g^2L}{B^2X^2}. \quad \text{(B1-42)}$$

v. At the low point of the oscillation the magnetic energy will be maximal, or

$$t_{\max} = \frac{T}{2} = \frac{\pi\sqrt{mL}}{BX}. \tag{B1-43}$$

Question 2

a. The radiation power absorbed by the patch of area a is

$$\epsilon a I \tag{B2-1}$$

while the power radiated from the patch is

$$\epsilon a \sigma T^4. \tag{B2-2}$$

The imbalance causes a temperature change given by

$$ca\frac{dT}{dt} = \epsilon a I - \epsilon a \sigma T^4, \tag{B2-3}$$

$$\frac{c}{\epsilon}\frac{dT}{dt} = I - \sigma T^4. \tag{B2-4}$$

b. We will need to do an expansion of T^4

$$T^4 = {T_0}^4\left(1 + \frac{T_1}{T_0}\sin(\omega t - \phi)\right)^4, \tag{B2-5}$$

$$\approx {T_0}^4\left(1 + 4\frac{T_1}{T_0}\sin(\omega t - \phi)\right). \tag{B2-6}$$

Insert this and the given expression for I into Eq. B2-4.

$$\frac{c}{\epsilon}\left(T_1\omega\cos(\omega t - \phi)\right) = \tag{B2-7}$$

$$I_0 + I_1\sin(\omega t) - \sigma {T_0}^4\left(1 + 4\frac{T_1}{T_0}\sin(\omega t - \phi)\right), \tag{B2-8}$$

or

$$\sigma {T_0}^4 + 4\sigma {T_0}^3 T_1 \sin(\omega t - \phi) + \frac{c\omega T_1}{\epsilon}\cos(\omega t - \phi) = I_0 + I_1\sin(\omega t). \tag{B2-9}$$

Clearly, then

$$A = \sigma {T_0}^4, \tag{B2-10}$$

$$B = 4\sigma {T_0}^3 T_1, \tag{B2-11}$$

$$C = \frac{c\omega T_1}{\epsilon} \tag{B2-12}$$

c. The only way that it can work is if $A = I_0$. Of course, one would expect the average intensity to be related to the average temperature in this way..

Expand the left hand sine and cosine terms according to the angle addition formulae:

$$\sin(\omega t - \phi) = -\cos\omega t\,\sin\phi + \sin\omega t\,\cos\phi, \tag{B2-13}$$

$$\cos(\omega t - \phi) = \cos\omega t\,\cos\phi + \sin\omega t\,\sin\phi \tag{B2-14}$$

Substitute into Equation (1) from the question paper.

$$B\left(-\cos\omega t\ \sin\phi + \sin\omega t\ \cos\phi\right) + C\left(\cos\omega t\ \cos\phi - \sin\omega t\ \sin\phi\right) = I_1 \sin\omega t. \qquad \text{(B2-15)}$$

This, after collecting terms, yields

$$(-B\sin\phi - C\cos\phi)\cos\omega t + (B\cos\phi + C\sin\phi)\sin\omega t = I_1\sin\omega t. \qquad \text{(B2-16)}$$

The cosine term must vanish, and this only happens if

$$-B\sin\phi + C\cos\phi = 0, \qquad \text{(B2-17)}$$

leaving the sine term, so

$$B\cos\phi + C\sin\phi = I_1. \qquad \text{(B2-18)}$$

Combining,

$$B\cos\phi + B\frac{\sin\phi}{\cos\phi}\sin\phi = I_1, \qquad \text{(B2-19)}$$

$$B = I_1\cos\phi. \qquad \text{(B2-20)}$$

and then

$$C = B\frac{\sin\phi}{\cos\phi} = I_1\sin\phi. \qquad \text{(B2-21)}$$

d. Evaluate B/A.

$$\frac{B}{A} = \frac{I_1}{I_0}\cos\phi, \qquad \text{(B2-22)}$$

and

$$\frac{B}{A} = \frac{4\sigma {T_0}^3 T_1}{\sigma {T_0}^4} = 4\frac{T_1}{T_0}, \qquad \text{(B2-23)}$$

we have

$$\frac{I_1}{I_0}\cos\phi = 4\frac{T_1}{T_0}. \qquad \text{(B2-24)}$$

Since $I_1/I_0 = 0.126/0.244 = 0.516$ we ought be concerned that T_1/T_0 might not be a sufficiently small quantity for this approximation!

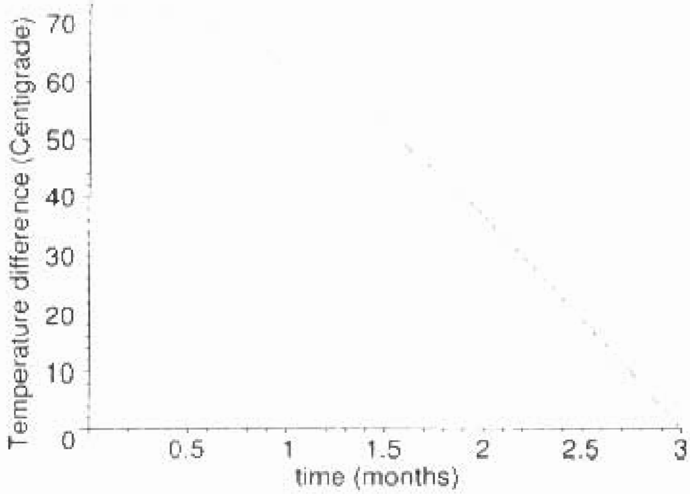
e. Since the temperature difference is *twice* the oscillation amplitude,

$$\Delta T = 2T_1, \qquad \text{(B2-25)}$$

so we are interested in graphing

$$\Delta T = \frac{T_0}{2}\frac{I_1}{I_0}\cos\phi. \qquad \text{(B2-26)}$$

$$= (73.1\ \text{K})\cos\left(t\frac{\pi}{6\ \text{month}}\right). \qquad \text{(B2-27)}$$



f. According to the data on the question paper the peak temperature happens about one month after the solstice. The predicted summer/winter temperature difference would then be about 62 C°, which is about 2.5 times that seen in the average temperature data for BWI.